Greek Symbols

Φ = angle of inclination of drop surface to upper plate measured through ambient fluid

ρ = density

Subscripts

b = refers to bottom plate

e = refers to equator of drop

h = refers to heavy fluid

l = refers to light fluid

t = refers to top plate

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Damping Coefficient Design Charts for Sampled-Data Control of Processes with Deadtime

Design charts that give the value of the gain of a proportional sampled-data controller for various closed loop damping coefficient specifications are presented. Charts for first- and second-order processes with various dead-times are given over a range of sampling rates.

Typical root locus plots in the z plane are also presented to illustrate how increasing deadtime (as an integer multiple of the sampling period) increases the order of the system in the z domain.

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SCOPE

The objective of this paper is to provide charts that the control systems designer can use to tune proportional sampled-data controllers.

With the rapidly increasing numbers of chromatographic control loops and digital process control computers, particularly the recent surge is minicomputers, chemical engineers must deal with sampled-data control systems more and more frequently. Straight-forward, easy-to-use charts should help to simplify the design and tuning of these sampled-data loops.

Previous workers have dealt extensively with sampled-data theory and mathematics (Tou, 1959), but applications-oriented papers have been few. The dependence of the maximum stable gain or the so-called "ultimate gain" on deadtime and sampling period has been studied (Mosler et al., 1966) for first-order processes. These authors also present curves for the gain that gives quarter-decay ratio performance. Designers also commonly use other specifications, such as the frequency-domain specifications of gain and phase margins and maximum closed loop log modulus (Luyben, 1971) or the time-domain specification of closed loop damping coefficient. The latter is employed in this paper.

CONCLUSIONS AND SIGNIFICANCE

Simple charts can be easily used to design sampled-data control loops for a desired closed loop damping coefficient specification. Increasing deadtime or sampling rate reduces the allowable gain for a given damping coefficient. However, for a given process with a fixed deadtime, the controller gain may be increased in some cases by decreasing the sampling rate or increasing the sampling period.

SYSTEMS STUDIED

Figure 1 gives a block diagram of the sampled-data system considered. The process consists of a first- or second-order lag $G_{M(s)}$ and a deadtime D, which is an integer multiple k of the sampling period T_s. The sampled-data controller $D_{(z)}$ has proportional-only action. The discontinuous output of $D_{(z)}$ is fed into a zero-order hold $H_{(s)}$ whose output M is held constant between samples.

The closed loop characteristic equations of these systems are:

First Order

$$1 + A_{(z)} = 1 + D_{(z)} Z[H_{(s)} G_{M(s)}]$$

$$= 1 + D_{(z)} Z \left[\left(\frac{1 - e^{-T_s s}}{s} \right) \left(\frac{e^{-Ds}}{\tau s + 1} \right) \right]$$

$$= 1 + K_c (1 - z^{-1}) z^{-k} Z \left[\frac{1}{s(\tau s + 1)} \right]$$

$$1 + A_{(z)} = 1 + K_c z^{-k} \left(\frac{1 - b}{z - b} \right) = 0 \quad (1)$$

$$z^k (z - b) + K_c (1 - b) = 0 \quad (2)$$

where τ = process time constant

k = D/T_s, an integer (3)

b = $\exp(-T_s/\tau)$ (4)

K_c = controller gain

Second Order

$$1 + A_{(z)} = 1 + D_{(z)} Z \left[\left(\frac{1 - e^{-T_s s}}{s} \right) \left(\frac{e^{-Ds}}{(\tau_1 s + 1)(\tau_2 s + 1)} \right) \right]$$

$$= 1 + K_c (1 - z^{-1}) z^{-k} Z \left[\frac{1}{s(\tau_1 s + 1)(\tau_2 s + 1)} \right]$$

$$1 + A_{(z)} = 1 + K_c z^{-k} a_0 \frac{(z + a_1)}{(z - p_1)(z - p_2)} = 0 \quad (5)$$

$$z^k (z - p_1)(z - p_2) + K_c a_0 (z + a_1) = 0 \quad (6)$$

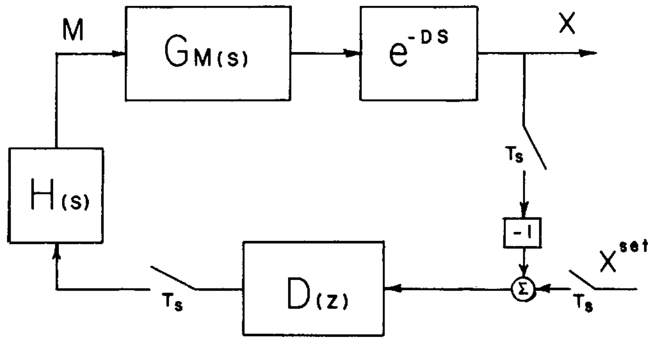

Fig. 1. Block diagram of sampled-data control system.

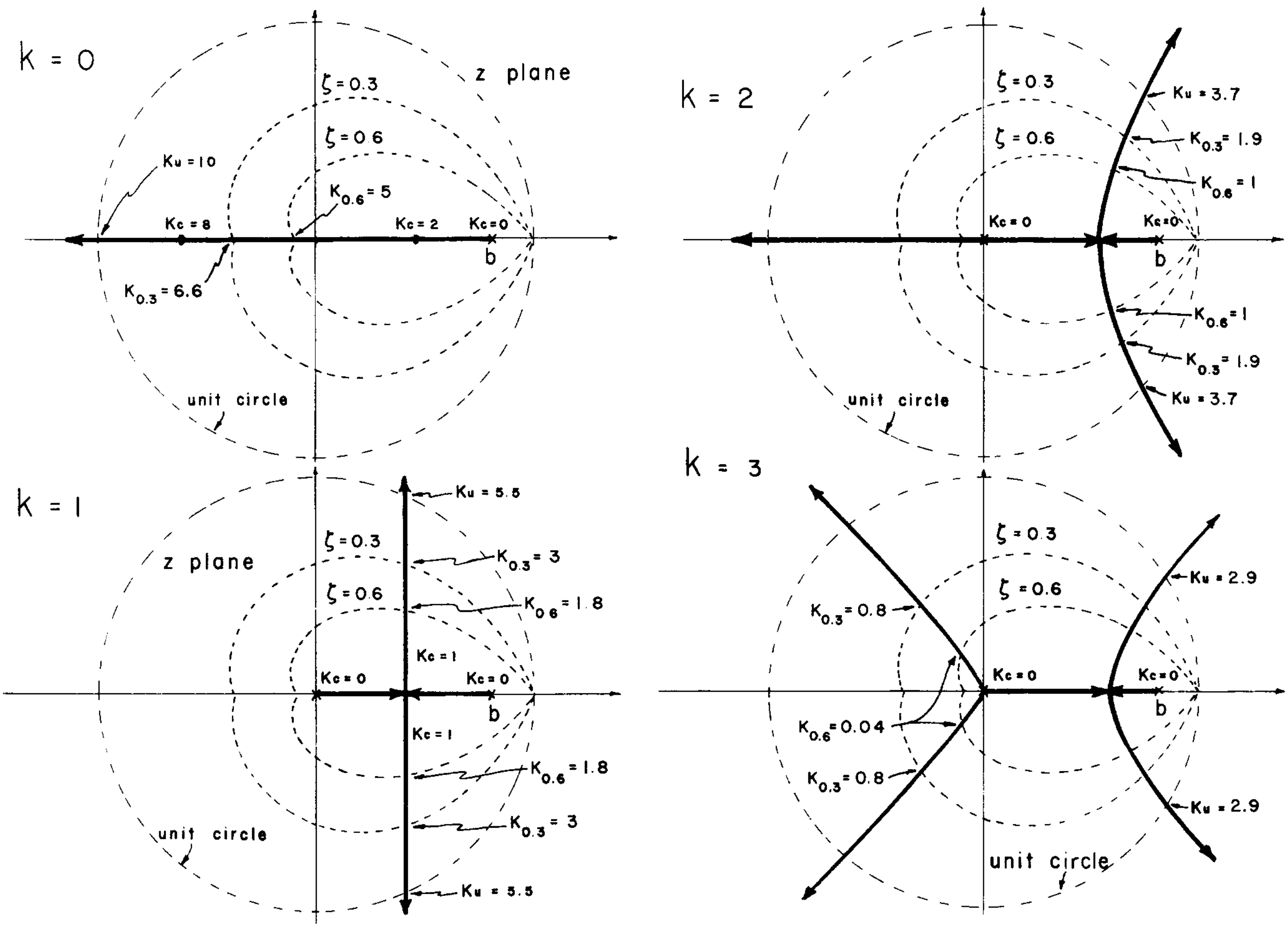

Fig. 2. Root locus plots of first-order system for several values of deadtime parameter $k = D/T_s$ with $T_s/\tau = 0.2$.

where τ_1, τ_2 = process time constants

$$p_1 = \exp(-T_s/\tau_1) \quad (7)$$

$$p_2 = \exp(-T_s/\tau_2) \quad (8)$$

$$a_0 = 1 + \frac{p_2 \tau_2 - p_1 \tau_1}{\tau_1 - \tau_2} \quad (9)$$

$$a_1 = \frac{p_1 p_2 (\tau_1 - \tau_2) + \tau_2 p_1 - \tau_1 p_2}{\tau_1 - \tau_2 + p_2 \tau_2 - p_1 \tau_1} \quad (10)$$

Notice that the order of the system is $(k + 1)$ for the first-order process and $(k + 2)$ for the second-order process. Increasing the deadtime for a given sampling rate increases the order of the system. Or decreasing the sampling period increases the order of the system for a given deadtime. This increase in order makes the system more difficult to control, and therefore control may not always be improved by reducing the sampling period. This suggests that a process with deadtime may be better controlled by a sampled-data controller with a finite sampling rate than by a continuous controller.

ROOT LOCUS PLOTS

Figures 2 and 3 show typical root locus plots for first- and second-order processes for several values of k. The sampling period for all these root locus plots was $T_s/\tau_1 = 0.2$. The ratio of time constants in the second-order system of Figure 3 was $\tau_2/\tau_1 = 0.25$.

Lines of constant damping coefficient ζ are sketched in these figures. They are given by the equation

$$z = \exp\left(\frac{-\zeta\,\omega\,T_s}{\sqrt{1-\zeta^2}}\right) \exp\,(i\,\omega\,T_s) \quad (11)$$

as ω varies from 0 to $\omega_s/2$.

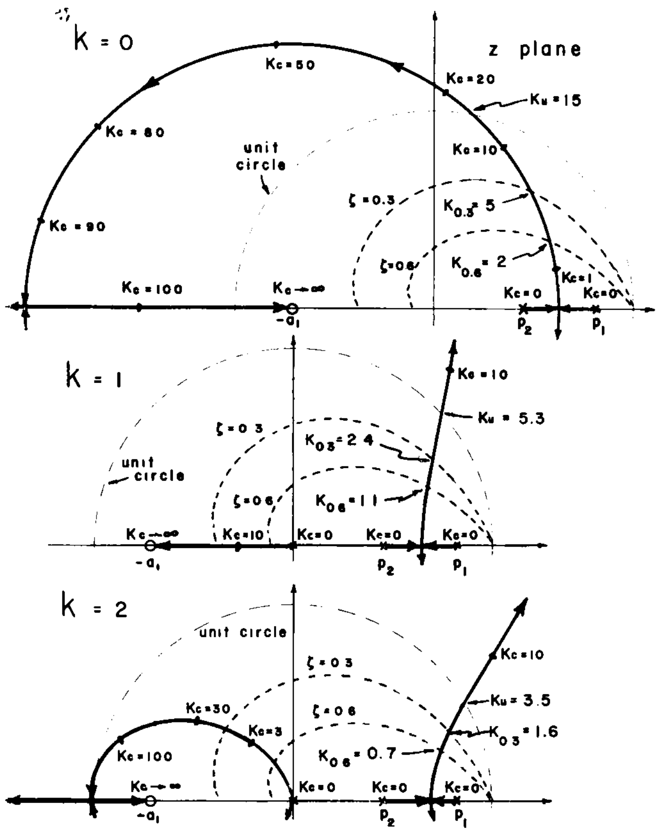

Fig. 3. Root locus plots of second-order system for several values of deadtime parameter $k = D/T_s$ with $T_s/\tau_1 = 0.2$ and $\tau_2/\tau_1 = 0.25$.

First Order (Figure 2)

When there is no deadtime $(k = 0)$, the first-order process has one root.

$$z = b - K_c\,(1 - b) \quad (12)$$

It moves to the left on the real axis from $+b$ to minus infinity as K_c varies from zero to infinity. The root intersects the damping coefficient lines on the negative real axis. This shows that, unlike a continuous first-order system, a first-order sampled-data system can be underdamped and closed loop unstable. The ultimate gain occurs when the path leaves the unit circle at $z = -1$.

$$K_u = \frac{1+b}{1-b} \quad (13)$$

When the deadtime D is equal to one sampling period T_s, k is equal to unity and the closed loop characteristic equation is

$$z^2 - bz + K_c\,(1 - b) = 0 \quad (14)$$

whose roots are

$$z = \frac{b}{2} \pm i\frac{1}{2}\sqrt{4\,K_c\,(1-b) - b^2} \quad (15)$$

There are now two paths which start $(K_c = 0)$ at $z = b$ and $z = 0$. They intersect the damping coefficient lines in the right-half plane. The ultimate gain is

$$K_u = \frac{1}{1-b} \quad (16)$$

For k's of 2 and 3 the closed loop characteristic equations are

$$z^3 - b\,z^2 + K_c\,(1 - b) = 0 \quad (17)$$

$$z^4 - b\,z^3 + K_c\,(1 - b) = 0 \quad (18)$$

The number of loci increases to three and four, respectively, with one path beginning $(K_c = 0)$ at $+b$ and the others beginning at the origin. The intersections with the damping coefficient lines can occur in either the right- or left-half plane. The ultimate gain occurs when the first complex path crosses the unit circle. The polynomial root-finding subroutine "POLRT" from the IBM Scientific Subroutine Library was used to solve for the roots of the closed loop characteristic equation at each value of gain.

Second Order (Figure 3)

Only the upper halves of the z planes are shown in Figure 3 to conserve space. The lower halves are merely reflections of the upper halves over the real axis.

When k is zero the closed loop characteristic equation is

$$z^2 + (K_c\,a_0 - p_1 - p_2)\,z + (p_1\,p_2 + a_0\,a_1\,K_c) = 0 \quad (19)$$

There are two paths which start at the two poles p_1 and p_2 of $A_{(z)}$ in Equation (5) and end at minus infinity and the zero $z = -a_1$. The damping coefficient lines are intersected when the roots are complex. The ultimate gain occurs when the loci leave the unit circle for small values of T_s/τ_1. For large values of T_s/τ_1 the complex part of the loci lies inside the unit circle, so the ultimate gain occurs where one of the real roots leaves the unit circle on the negative real axis.

For k's of 1 and 2, the closed loop characteristic equations are

$$z^3 + (-\,p_1 - p_2)\,z^2 + (p_1\,p_2 + K_c\,a_0)\,z + a_0\,a_1\,K_c = 0 \quad (20)$$

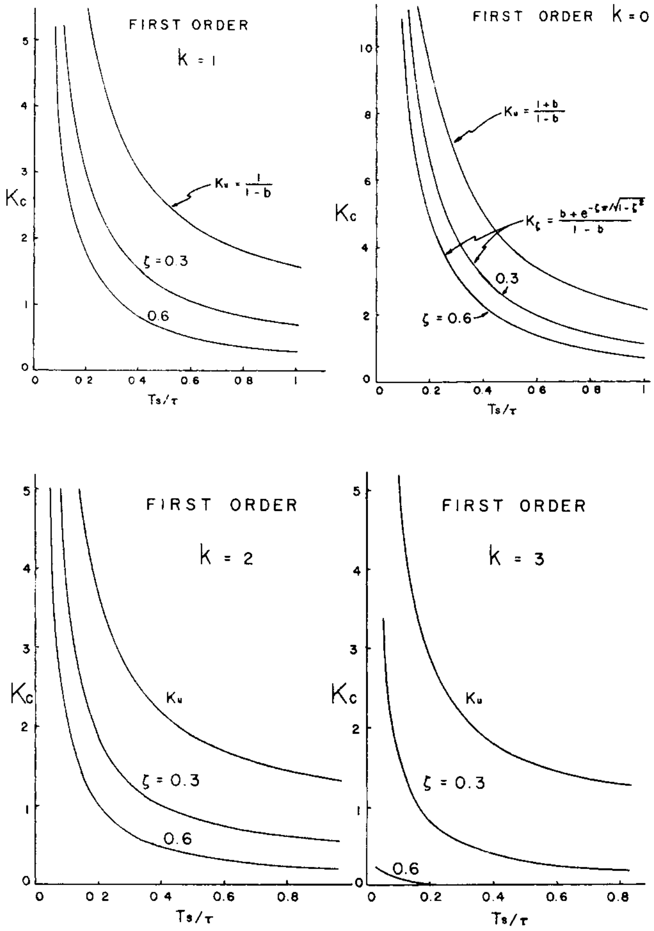

Fig. 4. Design charts for first-order systems.

$$z^4 + (-p_1 - p_2)\, z^3 + (p_1 p_2)\, z^2 + (a_0 K_c)\, z + a_0 a_1 K_c = 0 \quad (21)$$

There are now 3 and 4 paths, respectively.

DAMPING COEFFICIENT DESIGN CHARTS

The values of z that represent a line of constant damping coefficient ζ are given in Equation (11). The intersection of the root locus path with a constant damping coefficient line can be found by substituting Equation (11) into the characteristic equation of the system and solving for the value(s) of K_c and ωT_s.

First Order (Figure 4)

1. $k = 0$: Equating Equations (12) and (11) gives

$$z = b - K_c(1-b) = \exp\left(\frac{-\zeta\,\omega\, T_s}{\sqrt{1-\zeta^2}}\right) \exp\,(i\,\omega\, T_s)$$

Since there is only one real root, the intersection must occur on the negative real axis where $\omega = \omega_s/2 = \pi/T_s$. Therefore, $\omega T_s = \pi$ and $\exp(i\,\omega\, T_s)$ equals -1. Thus the value of gain K_ζ required to give any damping coefficient ζ is

$$K_\zeta = \frac{b + \exp(-\pi\,\zeta/\sqrt{1-\zeta^2})}{1-b} \qquad (22)$$

For the limit of stability where the damping coefficient is zero, this equation reduces to the same ultimate gain as Equation (13).

2. $k = 1$: Substituting Equation (11) into Equation (15) gives

$$[b/2] + i\left[\frac{1}{2}\sqrt{4\,K_c(1-b) - b^2}\right] = \left[\exp\left(\frac{-\zeta\,\omega\, T_s}{\sqrt{1-\zeta^2}}\right) \cos\,(\omega T_s)\right] + i\left[\exp\left(\frac{-\zeta\,\omega\, T_s}{\sqrt{1-\zeta^2}}\right) \sin\,(\omega T_s)\right]$$

Equating real parts gives an equation that can be solved (numerically) for $\omega\, T_s$ (between 0 and $\pi/2$). Then equating imaginary parts gives the value of gain. The results

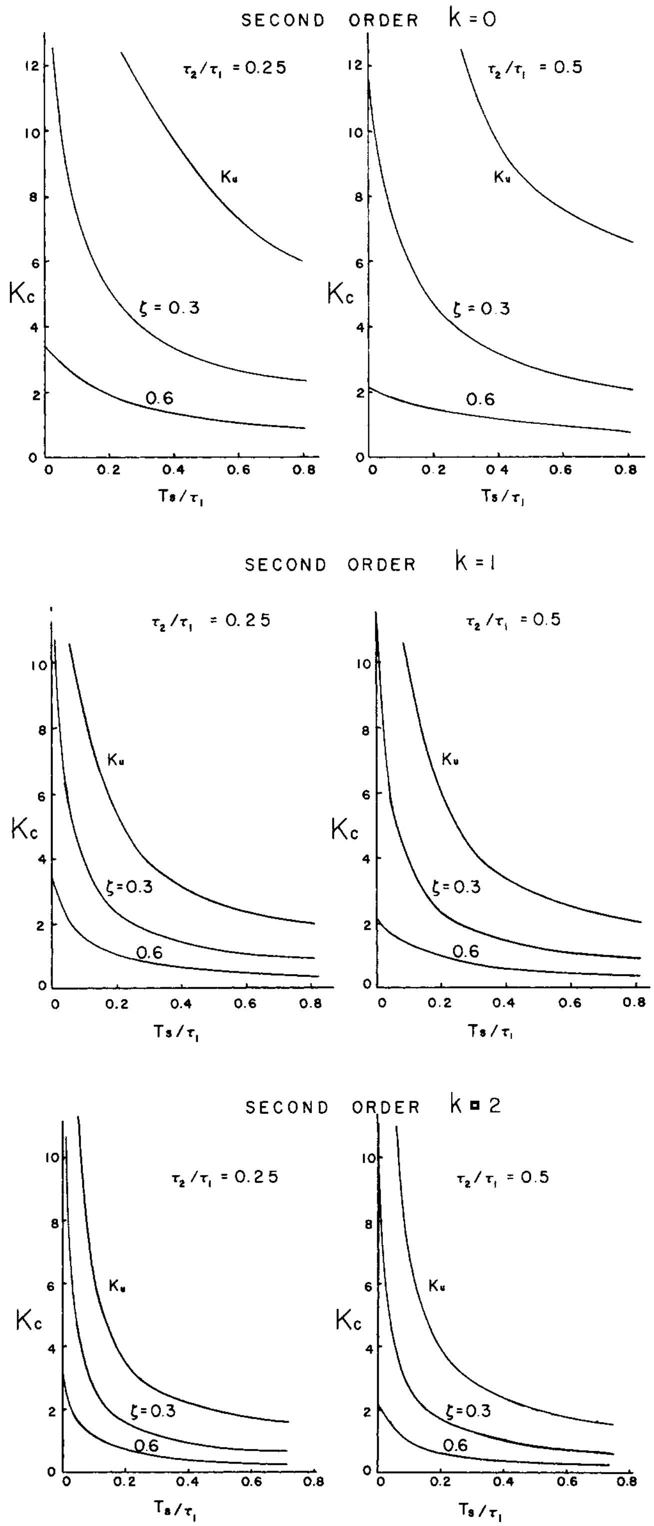

Fig. 5. Design charts for second-order systems.

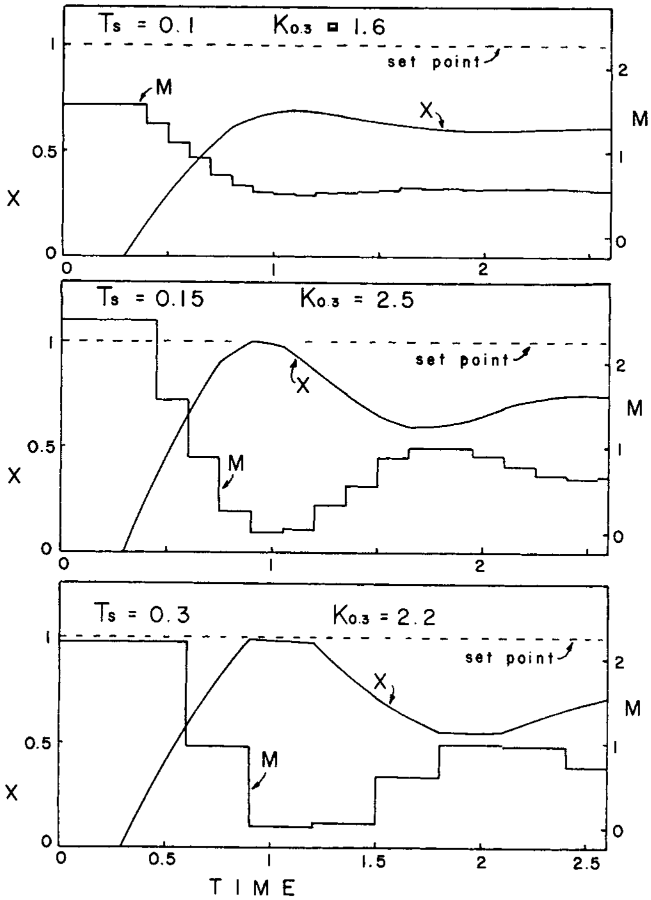

Fig. 6. Responses to a unit step change in set point with three sampling periods and corresponding $K_{0.3}$'s.

of these calculations are given in Figure 4 for damping coefficients of 0, 0.3, and 0.6.

3. $k = 2$ and $k = 3$: The numerical procedure described above was used for these cases. The equations are more complex algebraically, and the possibility of multiple roots must be considered.

Second Order (Figure 5)

The intersections of the damping coefficient lines and the root locus paths could have been found using the procedure described above. However for these more complex systems a direct numerical search for K_c was found more convenient. Simple interval halving was used to find the value of gain needed to give a root lying on the specified damping coefficient line.

The design curves for 0.3 and 0.6 damping coefficients in Figure 5 intersect the ordinate at finite values of gain as T_s/τ_1 goes to zero. Since deadtime D is equal to kT_s, D goes to zero as T_s goes to zero. Thus the system reduces to a continuous second-order system with no deadtime. The gain values for the two cases are given below.

For continuous systems
$(T_s/\tau_1 = D = 0)$

τ_2/τ_1	$K_{0.6}$	$K_{0.3}$
0.25	3.34	16.4
0.50	2.1	11.5

EXAMPLE

To illustrate the use of these charts, let us consider a first-order process with a 1 min. time constant ($\tau = 1$) and a 0.3 min. deadtime ($D = 0.3$). If a sampling period of 0.1 min. is used ($T_s = 0.1$), the ratio of D to T_s is 3.

$$k = D/T_s = 3$$

The gain for a 0.3 damping coefficient for $T_s/\tau = 0.1$ and $k = 3$ is found in Figure 4 to be $K_{0.3} = 1.6$.

If the sampling period is changed to 0.15 min. for the same process, k becomes $0.3/0.15 = 2$. The 0.3 damping coefficient gain for $T_s/\tau = 0.15$ and $k = 2$ is found from Figure 4 to now be $K_{0.3} = 2.5$. Notice that this is a higher gain than the gain with the 0.1 min. sampling period.

A further increase in T_s to 0.3 min. makes $k = 1$ and $T_s/\tau = 0.3$ and gives a $K_{0.3} = 2.2$. Therefore the 0.15 sampling period is better than the other two for this process. Figure 6 compares the responses of the system to a unit step change in set point with the three different sampling rates and $K_{0.3}$ gains.

It should be noted that the analysis above is limited to deadtimes that are integer multiples of the sampling period. It is felt that this is not an overly restrictive limitation in most practical problems. However modified z transforms could be used to generate similar charts for deadtimes that are noninteger multiples of the sampling period.

NOTATION

- $A_{(z)}$ = pulse transfer function of openloop system and controller
- a_0 = constant in $A_{(z)}$ transfer function
- a_1 = zero of $A_{(z)}$
- b = $\exp(-T_s/\tau)$
- D = deadtime
- $D_{(z)}$ = pulse transfer function of sampled-data controller
- $G_{M(s)}$ = process openloop transfer function
- $H_{(s)}$ = transfer function of zero-order hold
- k = ratio of deadtime D to sampling period T_s (an integer)
- K_c = controller gain
- K_u = ultimate gain of controller
- K_ζ = controller gain that gives a damping coefficient of ζ
- M = manipulative variable
- p_1 = pole of $A_{(z)}$ for second-order process
- p_2 = pole of $A_{(z)}$ for second-order process
- s = Laplace transform variable
- T_s = sampling period
- X = process output variable
- X^{set} = set point
- z = z transform variable
- Z[ ] = denotes z transformation
- ω = frequency (radians/time)
- ω_s = sampling frequency = $2\pi/T_s$ (radians/time)
- ζ = damping coefficient of closed loop system
- τ = process first-order time constant
- τ_1 = process time constant of second-order system
- τ_2 = smaller of the two second-order time constants

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